Stochastic analysis of induction machines using generalized polynomial chaos

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Abstract
It is well known that parametric and loading uncertainties can have adverse effects on the control of electric machines (Bose 1986). Machines used for ship propulsion, for example, are subject to random loading as a result of the seaway. To date, there have been few studies which can account for these stochastic effects in a comprehensive way. The generalized Polynomial Chaos (gPC)(Ghanem and Spanos 1991),(Lucor, Su and Karniadakis 2004),(Xiu and Karniadakis 2002),(Wan and Karniadakis 2005) is applied to induction machines in this paper. We show that the gPC technique captures the statistics of the response more efficiently than does Monte Carlo simulation.

Introduction
Many research projects have been conducted on various drive schemes for electric machines, to obtain robust performance against uncertainty, using either adaptation or estimation (Ortega et al. 1998). Because of the complexity of machine-converter dynamics, however, stochastic analysis of these systems in the presence of parametric and load uncertainties has not been extensively explored, from either an analytical or numerical perspective.

Stochastic differential equations (SDE’s) are solved for statistical responses by two main approaches today: Monte Carlo simulation (often with collocation), and the spectral stochastic processes approach. Although Monte Carlo simulation is an attractive and very broad method for solving general SDE’s of large and complex systems, it requires expensive computations to get good accuracy, and provides no explicit bounds for the estimated responses. The traditional stochastic process method, characterized by joint probability density functions and analytically based on the power spectral density of autocorrelation and cross-correlation (Leon-Garcia 1994), is often used in the analysis of both discrete- and continuous-time systems. The difficulty of this method is that it is limited to the computation of the Gaussian processes and linear time-invariant or simple-nonlinear (Iyengar and Basak 2005) systems. Clearly for complex electrical systems, the capability of capturing significant nonlinearity is essential.

In contrast to Monte Carlo simulation, which requires a computation of stochastic solutions before calculating statistical moments, the generalized Polynomial Chaos (gPC) technique, based on orthogonal polynomials, yields an explicit functional relationship between random noise and stochastic solutions, represented in terms of a spectral decomposition. A probability density function (PDF) as well as all the moments of system variables can be computed directly. The gPC was first introduced in the area of stochastic mechanics (Ghanem and Spanos 1991), and then extended to the time domain (Xiu and Karniadakis 2002). The major limitations associated with this technique derive from the complexity of expanding nonlinear and non-polynomial terms onto an orthogonal polynomial-chaos basis. This accounts for the trade-off between computational cost and accuracy, which is critical for large systems with many random parameters. A second-order linear oscillator (Lucor, Su and Karniadakis 2004) and the Kraichnan-Orszag threemode system (Wan and Karniadakis 2005) have been considered, as well as direct numerical simulation of fluid flows (Xiu et al. 2002).

In this study, the gPC is applied to examine the transient dynamic behavior of induction machines under variation of torque load and rotor resistance. The machines are subjected to a line-start, i.e., no controller is employed. While this condition is clearly unrealistic in practice, it forms a clean first application of the polynomial chaos approach in electric machines, and the basis for comparison with other methods. The stochastic time-variant inputs we consider are from a random process, modelled by the Karhunen-Loeve expansion (Leon-Garcia 1994). The sensitivity and stability of the propulsion system are investigated in terms of the mean
and variance solutions of all the states.

Modelling of the open-loop propulsion system is described in the next section. Then, we introduce the theory and implementation of generalized polynomial chaos. Stochastic analysis of the system follows.

**System Modeling**

The transient stability and the interaction between two induction machines are our main goals of this study. The open-loop configuration, based on (Mayer and Waszczyk 1991), is suitable to capture some of the important characteristics of real systems, although as noted the absence of a controller makes this an idealized case. See Figure 1; the physical parameters used in the study are given in Tables 1 and 2. We consider symmetrical three-phase squirrel-cage induction machines with three-phase windings in both the stator and rotor; the governing equations are written in the dq0 synchronous reference frame (denoted by superscript e). The voltage equations of the stator and rotor windings can be written as

\[
\begin{align*}
\psi^{e\,0}_{q0} &= -r_{s}\,i^{e\,0}_{q0} + \frac{\omega_{r}}{\omega_{b}} \left( T_{e} \psi^{e\,0}_{q0} + \frac{p}{\omega_{b}} \psi^{e\,0}_{q0} \right) \\
\psi^{e\,r} &= -r^{e\,r}_{q0} + \omega_{e} - \omega_{r} + \frac{p}{\omega_{b}} T_{1} \psi^{e\,r}_{q0} + \frac{p}{\omega_{b}} \psi^{e\,r}_{q0}
\end{align*}
\]

where \( \psi^{e\,0}_{q0} \) and \( \psi^{e\,r} \) are the stator and rotor variables of voltage, current, and flux, expressed in vector form as \( \mathbf{f}^{e\,0}_{q0} = [f^{e\,q}_{q0}, f^{e\,d}_{q0}, f^{e\,b}_{0}]^{T} \) and \( \mathbf{f}^{e\,r} = [f^{e\,q}_{q0}, f^{e\,d}_{q0}, f^{e\,b}_{0}]^{T} \), respectively. The resistance matrices \( r_{s} \) and \( r^{e\,r}_{q0} \) are diag[\( r_{s}, r_{s}, r_{s} \)] and diag[\( r^{e\,r}_{q0}, r^{e\,r}_{q0}, r^{e\,r}_{q0} \)]. \( \omega_{r} \) and \( \omega_{b} \) are the rotor angular and base velocities; the synchronous speed \( \omega_{e} \) is the same as \( \omega_{b} \) in the absence of a controller. The \( T_{1} \) matrix represents speed-voltage terms, with \( T_{1}(1, 2) = 1 \) and \( T_{1}(2, 1) = -1 \), and the \( p \) symbol represents the time derivative. The positive direction of stator current is assumed to be outward from the stator winding.

The equations of flux linkage per second are

\[
\begin{align*}
\psi^{e\,0}_{q0} &= -X_{ls} i^{e\,0}_{q0} + \psi^{e\,mqd} \\
\psi^{e\,r} &= X^{e\,r}_{l} i^{e\,r}_{q0} + \psi^{e\,mqd}
\end{align*}
\]

where the flux leakage matrices \( X_{ls} \) and \( X^{e\,r}_{l} \) are \( \text{diag}[x_{ls}, x_{ls}, x_{ls}] \) and \( \text{diag}[x^{e\,r}_{l}, x^{e\,r}_{l}, x^{e\,r}_{l}] \). We write \( \psi^{e\,mqd} = -X_{ls}^{e\,r} i^{e\,0}_{q0} + A_{r} \psi^{e\,0}_{q0} \), where \( X_{bs} = \text{diag}[x_{bs}, x_{bs}, x_{bs}] \) with \( x_{bs} = (x_{m1} x_{m1})/(x_{m1} + x_{m2}) \), and \( A_{r} = \text{diag}[x_{bs}/x_{m2}, x_{bs}/x_{m2}, 0] \). \( x_{ms} \) is the mutual inductance of the stator and rotor.

The dynamics of the mechanical subsystem can be written as

\[
p \omega_{r} = \frac{\omega_{b}}{2H} (T_{e} - T_{L}).
\]

where \( T_{e} = \psi^{e\,d_{s}} i^{e\,q_{s}} - \psi^{e\,d_{s}} i^{e\,q_{s}} \) and \( T_{L} \) are the electromagnetic and load torques, respectively. The rotor inertia (in seconds) is \( H \).

To couple two induction machines to the same electrical bus, causality of the voltage equations requires resolving into root and non-root machine models (Mayer and Waszczyk 1991). The root machine imposes the voltage on the bus, while the non-root machine provides the current to the bus. The state and output equations of the root induction machine are:

\[
\begin{align*}
p \, \psi^{e\,0}_{q0} &= \omega_{b} T_{1} \psi^{e\,0}_{q0} + \frac{\omega_{e} - \omega_{r}}{\omega_{b}} \psi^{e\,0}_{q0} + \frac{1}{\omega_{b}} \mathbf{X}_{s} p \psi^{e\,0}_{q0} + \frac{1}{\omega_{b}} \mathbf{A}_{r} p \psi^{e\,0}_{q0} \\
\psi^{e\,0}_{q0} &= -r_{s} i^{e\,0}_{q0} + \omega_{e} - \omega_{r} + \frac{p}{\omega_{b}} T_{1} \psi^{e\,0}_{q0} - \frac{1}{\omega_{b}} \mathbf{X}_{s} p \psi^{e\,0}_{q0} - \frac{1}{\omega_{b}} \mathbf{A}_{r} p \psi^{e\,0}_{q0}
\end{align*}
\]

The equations for the non-root induction machine are:

\[
\begin{align*}
p \, \psi^{e\,0}_{q0} &= \omega_{b} T_{1} \psi^{e\,0}_{q0} + \frac{\omega_{e} - \omega_{r}}{\omega_{b}} \psi^{e\,0}_{q0} + \frac{1}{\omega_{b}} \mathbf{X}_{s} p \psi^{e\,0}_{q0} + \frac{1}{\omega_{b}} \mathbf{A}_{r} p \psi^{e\,0}_{q0} \\
\psi^{e\,0}_{q0} &= -r_{s} i^{e\,0}_{q0} + \omega_{e} - \omega_{r} + \frac{p}{\omega_{b}} T_{1} \psi^{e\,0}_{q0} - \frac{1}{\omega_{b}} \mathbf{X}_{s} p \psi^{e\,0}_{q0} - \frac{1}{\omega_{b}} \mathbf{A}_{r} p \psi^{e\,0}_{q0}
\end{align*}
\]

where \( \mathbf{B}_{s} = (\mathbf{X}_{ls} + \mathbf{X}_{bs})^{-1} \) and \( \mathbf{B}_{r} = (\mathbf{X}_{ls})^{-1} \mathbf{A}_{r} \), using our previous definitions.
The coupled induction machines connect to an infinite bus through a series resistor-inductor tie-line; we use this specifically for studying the line-start transient dynamic in the open-loop configuration. The symmetrical three-phase tie-line can be modelled with the following equation:

\[ p \mathbf{i}_{q0}^e = \omega_b \mathbf{B}_t (\mathbf{v}_{q0}^e - \mathbf{r}_i \mathbf{i}_{q0}^e - \frac{\omega_e}{\omega_b} \mathbf{X}_t \mathbf{T}_i \mathbf{i}_{q0}^e). \]  

(10)

where \( r_i, L_t, \) and \( M_t \) are respectively resistance, inductance and mutual coupling inductance of the tie line; \( r_i = \text{diag}[r_{i1}, r_{i2}, r_{i3}], \mathbf{X}_t = \omega_b \text{diag}[L_t - M_t, L_t - M_t, L_t + 2M_t], \) and \( \mathbf{B}_t = X_t^{-1}. \)

### Table 1

<table>
<thead>
<tr>
<th></th>
<th>( r_s )</th>
<th>( x_{ls} )</th>
<th>( x_m )</th>
<th>( x_{lr} )</th>
<th>( x'_l )</th>
<th>( H )</th>
</tr>
</thead>
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<tr>
<td>IM1</td>
<td>0.01</td>
<td>0.0655</td>
<td>3.225</td>
<td>0.0655</td>
<td>0.0261</td>
<td>0.922</td>
</tr>
<tr>
<td>IM2</td>
<td>0.0051</td>
<td>0.00553</td>
<td>2.678</td>
<td>0.0553</td>
<td>0.0165</td>
<td>1.524</td>
</tr>
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</table>

### Table 2

<table>
<thead>
<tr>
<th>( r_l )</th>
<th>( L_t )</th>
<th>( M_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.01</td>
<td>0.004</td>
</tr>
</tbody>
</table>

### Application of Generalized Polynomial Chaos (gPC)

According to the Cameron-Martín theorem (Cameron and Martin 1947), the Wiener-Askey polynomial chaos expansion can approximate and describe all stochastic processes with finite second moment (variance); this is satisfied for most physical systems. The main idea is that all state variables of a random process can be expanded in terms of the weighted sum of orthogonal polynomials of standard random variables:

\[ \mathbf{x} = \sum_{i=0}^{\infty} \mathbf{x}_i(t) \phi_i(\zeta(\theta)), \]  

(11)

where \( \phi_i \) denotes a specific polynomial of the chaos, expressed as a function of a \( d \)-dimension random variable \( \zeta = (\zeta_1, ..., \zeta_d); \theta \) represents a random event confined within the sample space of the particular distribution. \( \mathbf{x}_i(t) \) is the \( i \)th modal coefficient, from which statistics can be directly calculated. In numerical implementation, the expansion onto orthogonal polynomial basis is truncated at \( P \) terms. We have \( P = (d + p)!/d!p! \), where \( d \) and \( p \) are the dimension of random variable \( \zeta \) and the highest order of the polynomial chaos, respectively.

An important property of the Wiener-Askey polynomial chaos is orthogonality:

\[ \langle \phi_i, \phi_j \rangle = \langle \phi_i^2 \rangle \delta_{ij} \]  

(12)

where \( \delta_{ij} \) is the Kronecker delta and \( \langle \cdot, \cdot \rangle \) represents the inner product on the support of random variable \( \zeta \). In the continuous case, the inner product is defined as:

\[ \langle f(\zeta), g(\zeta) \rangle = \int f(\zeta) \cdot g(\zeta) \cdot W(\zeta) d\zeta, \]  

(13)

where \( W(\zeta) \) is the weighting function corresponding to the chaos. In accordance with the Wiener-Askey scheme (Xiu and Karniadakis 2002), optimum performance is obtained by matching specific classes of orthogonal polynomials with specific distributions. The Hermite polynomial chaos is suitable for a Gaussian random process. In our case, we assume all the uncertainties to have uniform distribution, and thus the Legendre polynomial chaos is the most efficient basis.

Consider now the ODE of the induction machine mechanical subsystem. After expanding all the state variables onto the orthogonal polynomial basis with the uncertainty in torque load, the stochastic differential equation becomes

\[ \sum_{i=0}^{P} \frac{d\omega_{ri}}{dt} \phi_i = \frac{\omega_b x_b}{2H x'_{lr}} \left( \sum_{i=0}^{P} \sum_{j=0}^{P} q_{qi} \phi_i \sum_{j=0}^{P} q_{sj} \phi_j \right) \]  

(14)

\[ \frac{\omega_b x_b}{2H} \sum_{i=0}^{P} T_{Li} \phi_i, \]

\[ \frac{\omega_b}{2H} \sum_{i=0}^{P} T_{Li} \phi_i. \]

Projection the above equation onto each of the \( P + 1 \) modes of the polynomial chaos, using the Galerkin method, results in \( P + 1 \) deterministic ODE’s for each mode in the expansion:

\[ \frac{d\omega_{ri,k}}{dt} \langle \phi_k^2 \rangle = \frac{\omega_b x_b}{2H x'_{lr}} \sum_{i=0}^{P} \sum_{j=0}^{P} q_{qi} \phi_i \sum_{i=0}^{P} q_{ij} \phi_j \]  

(15)

\[ \frac{\omega_b}{2H} T_{Li} \langle \phi_k^2 \rangle. \]
where $\epsilon_{ijk}$ denotes the triple product $\langle \phi_i, \phi_j, \phi_k \rangle$. The rest of the SDE’s of the system are similarly transformed into deterministic ODE’s. For the random process case, coefficients of a low-order Karhunen-Loeve (K-L) expansion are assigned to random variables (Leon-Garcia 1994), (Lucor, Su and Karniadakis 2004). The K-L expansion assumes a priori knowledge about the covariance kernel; the exponential covariance kernel, which depends solely on the correlation length, is the focus in this study. Once the time integration of the modes is complete, first and second moments of each physical state can be directly obtained from the zero modes, and a summation of squared modal amplitudes, each multiplied by $\langle \phi_k^2 \rangle$. We give calculated results in the tables below also for the multi-element gPC method, which achieves substantially better performance (Wan and Karniadakis 2005). This method divides the domains of each random variable to allow the use of lower-order polynomials within each sub-element.

Results and Discussion

We investigate the dynamic effect of a random process acting on two different electric-machine systems: 1) the open-loop line start of a single induction machine (seven ODE’s), and 2) the open-loop line start of two coupled induction machines (fourteen ODE’s). As described in the section above, the systems are connected to an infinite bus through a tie line.

Open-Loop Line Start of a Single Induction Machine

The effect of rotor resistance fluctuation is examined for the start-up transient dynamic of the 200-hp root induction machine connecting with the tie line to the infinite bus. The fluctuation is realized as a three-term K-L expansion, illustrated in Figure 2. Hence the random dimension in all runs is three. The correlation length is ten seconds, and the mean and variance of rotor resistance are 0.0261 p.u. and 0.0001, respectively. For these runs, the torque load is held constant at the value of 0.5 p.u.

Both the first and second moments of the $dq$-axis rotor fluxes, per-unit angular velocity, and $dq$-axis tie-line and stator currents are shown in Figure 3. Note that all state variables are expressed in per unit, and that the 0-axis flux and current variables are uncoupled from the others.

Overall, the machine draws large current during its start-up transient, and the response reaches a steady-state value within three seconds. The high-frequency characteristic of states in the first one-half second is captured accurately by the gPC method, when compared to Monte Carlo results (see below). Figure 3 also shows both fast and slow transient dynamics corresponding to the electrical and mechanical subsystems; the slow response of the mechanical component contributes to a smooth variance in all the electrical states between one and two seconds, and all variances converge to zero after three seconds. This confirms that the system is stable when subjected to multiplicative uncertainty in the rotor resistance. The rotor resistance fluctuation induces about the same magnitude of variation in all electrical states, except that the variance of stator current in the $dq$-synchronous frame is ten times larger than that of the rotor current, and hence this is a point of increased sensitivity.

To check these results from the gPC and the MEgPC


techniques, a large-scale quasi-Monte Carlo (MC) simulation is used for comparison (Press et al. 1992). Quasi-MC techniques can achieve $1/n$-convergence, so eighty-thousand trials corresponds with 6.4e9 trials in a normal MC method. In this case, the mean values can be expected to be accurate to about one part in $10^6$. Table 1 provides the computational time and the $L_2$-norm difference for both mean and variance solutions via the gPC and MEgPC methods, relative to the quasi-MC results. The differences in the MEgPC case are consistent with the quasi-MC estimate, and hence the calculations achieve approximately the same level of accuracy. The gPC results are not as accurate and are substantially more expensive. The $L_2$-norm difference is defined as $||x_{gPC}(t) - x_{MC}(t)||/||x_{MC}(t)||$.

### Table 3: $L_2$-norm difference (nondimensional) of mean $(E)$ and variance $(\sigma^2)$ solutions for: 1) gPC $(p=15)$ and 2) MEgPC $(p=1,N=8000)$ methods, relative to a quasi-MC analysis (80,000 trials).

<table>
<thead>
<tr>
<th>L_2 diff</th>
<th>$\psi_{q r 1}$</th>
<th>$\psi_{d r 1}$</th>
<th>$\omega_{r 1}/\omega_b$</th>
<th>$i_{r 1}$</th>
<th>$i_{d 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 : $E$</td>
<td>$8.95e^{-3}$</td>
<td>$2.21e^{-3}$</td>
<td>$4.06e^{-5}$</td>
<td>$8.01e^{-4}$</td>
<td>$2.60e^{-4}$</td>
</tr>
<tr>
<td>1 : $\sigma^2$</td>
<td>$1.26e^{-2}$</td>
<td>$2.00e^{-3}$</td>
<td>$4.63e^{-4}$</td>
<td>$1.92e^{-2}$</td>
<td>$1.88e^{-4}$</td>
</tr>
<tr>
<td>2 : $E$</td>
<td>$3.47e^{-5}$</td>
<td>$1.05e^{-5}$</td>
<td>$2.56e^{-5}$</td>
<td>$2.86e^{-5}$</td>
<td>$1.13e^{-5}$</td>
</tr>
<tr>
<td>2 : $\sigma^2$</td>
<td>$3.85e^{-4}$</td>
<td>$7.92e^{-5}$</td>
<td>$7.56e^{-5}$</td>
<td>$3.60e^{-4}$</td>
<td>$7.46e^{-5}$</td>
</tr>
</tbody>
</table>

### Table 4: Computational cost for the quasi-MC (80,000 trials), gPC $(p=15)$, MEgPC $(p=1,N=8000)$ methods.

<table>
<thead>
<tr>
<th></th>
<th>gPC</th>
<th>MEgPC</th>
</tr>
</thead>
<tbody>
<tr>
<td>quasi-MC</td>
<td>355</td>
<td>1038</td>
</tr>
</tbody>
</table>

### Open-loop Line Start of Two Induction Machines

When two induction machines are connected to the same electrical bus, rotor resistance uncertainty in one of the machines can affect the transient behavior and stability of the entire system, as we now show. For this case, the variation of rotor resistance is the same as in the previous case, in the 200-hp machine; the second machine is rated at 150 hp. The base power of each machine is correspondingly equal to machine rating with a base rms, line-to-line voltage ($V_{base}$) = 450 V. The infinite bus’s base power is 3125 kVA with the same base voltage.

Figure 4 shows the mean stochastic solutions for the two motors (1 and 2 subscripts), and Figure 5 shows the variance solutions. We see that there is virtually no influence of this uncertainty on the 150-hp induction machine; the mean is largely the same as the deterministic case and the variance is extremely small. The response of the first machine is virtually identical to the case of the previous section, with very similar mean and variance solutions. The interaction between the two machines is very limited because both are directly connected to an infinite bus which can absorb some of the power fluctuations.

### Summary and Conclusion

We applied the polynomial chaos technique for stochastic simulation to a core nonlinear component in electric power systems, namely induction machines. Although our study is extremely simplified - no controllers are employed - the chaos approach can provide high-fidelity results with a significant cost savings over traditional Monte Carlo approaches. We confirm that these cost savings are enhanced for higher-order and high random dimension systems, in agreement with prior studies in mechanics and CFD. This suggests that the approach should find good application in complex electrical power systems. We plan to extend the technique to systems of realistic size, e.g., having hundreds of states, and containing controllers, generators, prime movers, and various dynamic loads. In addition, the polynomial
FIGURE 5. Comparison of the variance solution from stochastic simulation of two induction machines using two approaches: Monte Carlo simulation with 25,000 realizations, and polynomial chaos with $p = 15$. All state variables are per-unit.

chaos technique can be adapted for non-polynomial nonlinearities, including the signum function, and hence it should be suitable for switching components.

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References


